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Noncommutative Classical Mechanics

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In this work, I investigate the noncommutative Poisson algebra of classical observables corresponding to a proposed general noncommutative quantum mechanics, Djemai, A. E. F. and Smail, H. (2003). I treat some classical systems with various potentials and some physical interpretations are given concerning the presence of noncommutativity at large scales (celestial mechanics) directly tied to the one present at small scales (quantum mechanics) and its possible relation with UV/IR mixing.

KEY WORDS: noncommutative space; classical mechanics; Moyal product.

1. INTRODUCTION

It is well-known that quantum mechanics (QM) can be viewed as a noncommutative (matrix) symplectic geometry, (Djemai, 1996), generalizing the usual description of classical mechanics (CM) as a symplectic geometry.

In the context of the algebraic star–deformation theory, QM was also described as a $\hbar$–deformation of the algebra $\mathcal{A}_0$ of classical observables. The procedure consists to replace the operator algebra issued from standard quantization rules by the algebra $\mathcal{A}_\hbar$ of "quantum observables" generated by the same classical observables obeying actually a new internal law other than the usual point product, the so-called Moyal star-product, (Bayen *et al.*, 1977, 1978; Flato *et al.*, 1975, 1976; Moyal, 1949; Vey, 1975), such that the "classical" limit is guaranteed by $\hbar \to 0$. This is the program of "quantization by deformation" carried out by Lichnerowicz *et al.*

Moreover, in the lattice quantum phase space, (Djemai, 1995, 1996), the discretization parameter $\frac{2\pi}{N}$ can be interpreted as a deformation parameter. It is also well-known that, as the "classical" limit $\hbar \to 0$ ensures the passage from QM to CM, the passage, for instance, from relativistic CM to nonrelativistic CM is ensured by the "classical" limit $\beta = \frac{v}{c} \to 0$, where v is the velocity of the classical particle and c is the light velocity.

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Recently, there has been a big interest in the study of various physical theories: string theory, (Chu and Ho, 1999, 2000; Connes *et al.*, 1998; Douglas and Hull, 1998; Schomerus, 1999; Seiberg and Witten, 1999), quantum field theories, (Chaihian *et al.*, 2001, 2002; Szabo, 2001), QM, (Acatrinei, 2001; Bellucci *et al.*, 2001; Christiansen and Schaposnik, 2001; Gamboa et al., 2001a; Gamboa *et al.*, 2001b; Ho and Kao, 2001; Kochan and Demetrian, 2001; Morariu and Polychronakos, 2001; Nair and Polychronakos, 2001), condensed matter, (Ezawa, 2000), . . . , on noncommutative spaces. Furthermorre, the notion of noncommutativity may receive different Physical interpretations. The most particular one consists to do the parallel between the mechanics of a quantum particle in the usual space in presence of a magnetic field B and the mechanics of this quantum particle moving into a noncommutative space. Furthermore, SUSY, through its Z_2-graded algebra, may be viewed as a particular case of noncommutativity. This means that *superpartners* of ordinary quantum particles can be studied only if one considers a particular kind of noncommutativity, namely SUSY. Moreover, the deformation parameter seems to be a *fundamental constant* which characterizes the Physics described on a noncommutative space.

The aim of this work is, following the general formulation of noncommutative quantum mechanics (NCQM) proposed in Djemai and Smail (2003) and generalizing the approach of (Romero and Vergara, 2003; Mirza and Dehghani, 2002) to discuss the associated noncommutative classical mechanics (NCCM) and to treat some particular examples of classical potentials.

The work is organized as follows. Section 2 is devoted to a brief and methodic presentation of the passage from CM to QM, and from QM to NCQM in view to fix notations. In Section 3, I derive the associated NCCM and discuss different aspects concerning the star-deformed Poisson algebra and the resulting motion equations. In Section 4, I treat different cases of classical potentials $V(x)$ like the free particle, the harmonic oscillator and, in particular, the gravitational potential. The parallel between this latter classical case and Coulomb potential in QM is discussed. Finally, I devote Section 5 to some conclusions and perspectives.

2. CM → QM → NCQM

Let us first start by considering a *classical system* with an Hamiltonian:

$$H(x, p) = \frac{p^2}{2m} + V(x) \tag{1}$$

where the coordinates x_i and the momenta p_i, $i = 1, \ldots, N$, generate the algebra $\mathcal{A}_0$ over the Classical Phase Space (CPS) with the usual Poisson structure:

$$\{x_i, x_j\}_p = 0, \quad \{x_i, p_j\}_p = \delta_{ij}, \quad \{p_i, p_j\}_p = 0$$

or in terms of phase space variables u_a, $a = 1, \ldots, 2N$:

$$\{u_a, u_b\} = \omega_{ab}$$

where ω is called the *classical symplectic structure* and is represented by the $2N \times 2N$ matrix:

$$\omega = \begin{pmatrix} 0 & \mathbf{1}_{N\times N} \\ -\mathbf{1}_{N\times N} & 0 \end{pmatrix}$$

with $Det(\omega) = 1$.

Moreover, the motion equations of the classical system are given by:

$$\dot{x}_i = \{x_i, H\}, \quad \dot{p}_i = \{p_i, H\}.$$

Now, consider a *Dirac quantization* of this system:

$$\{f, g\}_P \to \frac{1}{i\hbar}[\mathcal{O}_f, \mathcal{O}_g]$$

where we denote by $\mathcal{O}_f$ the operator associated to a classical observable f, with, in particular, $\mathcal{O}_{x_i} = \mathbf{x}_i$ and $\mathcal{O}_{p_i} = \mathbf{p}_i$. These operators generate the Heisenberg algebra:

$$[\mathbf{x}_i, \mathbf{x}_j] = 0, \quad [\mathbf{x}_i, \mathbf{p}_j] = i\hbar\delta_{ij}\mathbf{1}, \quad [\mathbf{p}_i, \mathbf{p}_j] = 0.$$

Furthermore, the motion of this quantum system is governed by the *cononical equations*:

$$\dot{\mathbf{x}}_i = [\mathbf{x}_i, \mathbf{H}], \quad \dot{\mathbf{p}}_i = [\mathbf{p}_i, \mathbf{H}]$$

where:

$$\mathbf{H}(\mathbf{x}, \mathbf{p}) = \frac{\mathbf{p}^2}{2m} + V(\mathbf{x}).$$

It is well known also that this *quatization* is equivalent to a $\hbar$-star deformation of $\mathcal{A}_0$ such that the Heisenberg operator algebra is replaced by the algebra $\mathcal{A}_\hbar$:

$$\{x_i, x_j\}_\hbar = 0, \quad \{x_i, p_j\}_\hbar = i\hbar\delta_{ij}, \quad \{p_i, p_j\}_\hbar = 0 \tag{2}$$

generated by the same classical observables but now obeying a Moyal product:

$$(f \star_\hbar g)(u) = \exp\left[\frac{i}{2}\hbar\omega^{ab}\partial_a^{(1)}\partial_b^{(2)}\right] f(u_1)g(u_2)|_{u_1=u_2=u}$$

where:

$$\omega^{ab}\omega_{bc} = \delta^a_c$$

and

$$\{f, g\}_{\hbar} = f \star_{\hbar} g - g \star_{\hbar} f.$$

Let us now consider another α-star deformation of the algebra $\mathcal{A}_0$, such that the internal law will be characterized not only by the *fundamental constant* $\hbar$ but also by another deformation parameter (or more). This can be performed by generalizing the usual symplectic structure into another more general one, say α_{ab}. For instance, let us consider the algebra $\mathcal{A}_\alpha$ equipped with the following star-product, (Djemai and Smail, 2003):

$$(f \star_{\hbar,\alpha} g)(u) = \exp\left[\frac{i\hbar}{2}\alpha^{ab}\partial_a^{(1)}\partial_b^{(2)}\right] f(u_1)g(u_2)|_{u_1=u_2=u} \tag{3}$$

such that

$$\alpha_{ab} = \begin{pmatrix} \theta_{ij} & \delta_{ij} + \sigma_{ij} \\ -\delta_{ij} - \sigma_{ij} & \beta_{ij} \end{pmatrix}$$

where the $N \times N$ matrices θ and β are assumed to be antisymmetric:

$$\theta_{ij} = \epsilon_{ij}{}^k\theta_k, \quad \beta_{ij} = \epsilon_{ij}{}^k\beta_k$$

while σ is assumed to be symmetric and it will be neglected since it is of second order, (Djemai and Smail, 2003). This new star-product generalizes the relations (2) in the following way:

$$\{x_i, x_j\}_{\hbar,\alpha} = i\hbar\theta_{ij}, \quad \{x_i, p_j\}_{\hbar,\alpha} = i\hbar(\delta_{ij} + \sigma_{ij}), \quad \{p_i, p_j\}_{\hbar,\alpha} = i\hbar\beta_{ij}$$

and so gives rise to a NCQM defined by the following generalized Heisenberg operator algebra:

$$[\mathbf{x}_i, \mathbf{x}_j]_\alpha = i\hbar\theta_{ij}\mathbf{1}, \quad [\mathbf{x}_i, \mathbf{p}_j]_\alpha = i\hbar(\delta_{ij} + \sigma_{ij})\mathbf{1}, \quad [\mathbf{p}_i, \mathbf{p}_j]_\alpha = i\hbar\beta ij\mathbf{1}.$$

In (Djemai and Smail, 2003), we have found that the matrix σ is tied to the anticommutator of θ with β, and that the determinent of the matrix α is given in function of $\rho = Tr(\theta\beta) = Tr(\beta\theta)$. If we impose to the determinent to be equal to 1, then one obtains that:

$$\rho = -2\vec{\theta}\cdot\vec{\beta}$$

which is deeply linked to the Heisenberg Incertitude relations.

3. NONCOMMUTATIVE CLASSICAL MECHANICS

The purpose of this paper is precisely to study the *noncommutative classical mechanics* which leads to the NCQM as described in the previous section. The passage between NCCM and NCQM is assumed to be realized via the following *generalized Dirac quantization*:

$$\{f, g\}_\alpha \to \frac{1}{i\hbar}[\mathcal{O}_f, \mathcal{O}_g]_\alpha.$$

It follows that our *Noncommutative classical Mechanics* is described by the α-star deformed classical Poisson algebra $\mathcal{A}_\alpha$ generated by the classical position and momentum variables obeying to this new internal law, namely (3) without $i\hbar$:

$$(f \star_\alpha g)(u) = \exp\left[\frac{1}{2}\alpha^{ab}\partial_a^{(1)}\partial_b^{(2)}\right] f(u_1)g(u_2)|_{u_1=u_2=u}$$

such that:

$$\{x_i, x_j\}_\alpha = \theta_{ij}, \quad \{x_i, p_j\}_\alpha = \delta_{ij} + \sigma_{ij}, \quad \{p_i, p_j\}_\alpha = \beta_{ij}. \tag{4}$$

Using the Hamiltonian (1), we get the following Hamilton's equations :

$$\dot{x}_i = \{x_i, H\}_\alpha = \frac{p_i}{m} + \theta_{ij}\frac{\partial V}{\partial x_j} + \frac{1}{m}\sigma_{ij}p^j \simeq \frac{p_i}{m} + \theta_{ij}\frac{\partial V}{\partial x_j} \tag{5}$$

$$\dot{p}_i = \{p_i, H\}_\alpha = -\frac{\partial V}{\partial x^i} + \frac{1}{m}\beta_{ij}p^j - \sigma_{ij}\frac{\partial V}{\partial x_j} \simeq -\frac{\partial V}{\partial x^i} + \frac{1}{m}\beta_{ij}p^j. \tag{6}$$

In the noncommutative configuration space, the classical particle obeys the following motion equations:

$$\begin{aligned} m\ddot{x}_i &= -\frac{\partial V}{\partial x^i} + m\theta_{ij}\left(\frac{\partial^2 V}{\partial x_k \partial x_j}\right) \star \dot{x}_k \\ &\quad + [(1+\sigma)\beta(1+\sigma)^{-1}]_{ik}\dot{x}_k \\ &\quad - [\sigma + (\mathbf{1}+\sigma)\sigma + (1+\sigma)\beta(\mathbf{1}+\sigma)^{-1}\theta]_{ik}\left(\frac{\partial V}{\partial x_k}\right) \\ &\simeq -\frac{\partial V}{\partial x^i} + \left[m\theta_{ij}\left(\frac{\partial^2 V}{\partial x_k \partial x_j}\right) + \beta_{ik}\right] \star \dot{x}_k \\ &\quad + O(\theta^2) + O(\beta^2) + O(\sigma). \end{aligned} \tag{7}$$

where $\mathbf{1}$ means the 3×3 unit matrix.

The first term in the right side of this equation, that can be obtained by taking the *classical limit* ($\theta = \beta = \mathbf{0}$), represents the usual expression of a conservative force which derives from a potential $V(x)$ present on the commutative space (The second Newton law). The second term, which has been found in Romero and Vergara (2003), expresses a first correction to this law depending on the presence of a noncommutativity only on the configuration space ($\theta \neq \mathbf{0}$ and $\beta = \mathbf{0}$) and also on the variations of the external potential $V(x)$, (Romero *et al.*, 2002). The third term, that is a kinetic correction term, reflects a second correction due to the presence of a noncommutativity only on the momentum sector of the classical phase space ($\theta = \mathbf{0}$ and $\beta \neq \mathbf{0}$).

Hence, this result is very general in the sense that it takes into account the noncommutativity on the whole phase space, since we have shown in (Djemai and Smail, 2003), that the presence of a noncommutativity on the configuration space characterized by the parameter θ will automatically imply the presence of a noncommutativity on the momentum sector characterized by the parameter β, such that the two parameters are subject, through the parameter ρ, to a lower bound constraint:

$$\rho = Tr[\theta\beta] = Tr[\beta\theta] = -2\vec{\theta}\cdot\vec{\beta} = -16.$$

Indeed, the two parameters may exist and vary simultaneously but are tied by the above constraint which has a direct physical interpretation (Heisenberg incertitude relations), (Djemai and Smail, 2003).

Moreover, we remark that, in addition to the classical first term in (7), there is an additional term given in terms of $\dot{x}_k$ that can be interpreted as the presence of some kind of *viscosity* (resistivity) in the phase space due to its noncommutativity property and also to the variations of the potential.

Let us now consider a particular transformation on the usual classical phase space (CPS) that leads to the same results as of the $\star_\alpha$–deformation on CPS, like the nontrivial commutation relations (4) or the motion equations (5), (6), or (7). Indeed, following the same approach as in (Djemai and Smail, 2003), we introduce the following transformation on usual CPS:

$$x_i' = x_i - \frac{1}{2}\theta_{ij} p_j, \quad p_i' = p_i + \frac{1}{2}\beta_{ij} x_j \tag{8}$$

Firstly, it is easy to check that:

$$\{x_i', x_j'\}_P = \theta_{ij}, \quad \{x_i', p_j'\}_P = \delta_{ij} + \sigma_{ij}, \quad \{p_i', p_j'\}_P = \beta_{ij}. \tag{9}$$

where the symmetric 3×3-matrix σ is given by:

$$\sigma = -\frac{1}{8}[\theta\beta + \beta\theta].$$

Then, the usual Poisson brackets give the following Hamilton's equations:

$$\dot{x}_i' = \{x_i', H'\}_P = \frac{p_i'}{m} + \theta_{ik}\frac{\partial V'}{\partial x_k'} \tag{10}$$

$$\dot{p}_i' = \{p_i', H'\}_P = -\frac{\partial V'}{\hat{\partial} x'^i} + \frac{1}{m}\beta_{ik} p'^k. \tag{11}$$

which looks like (5) and (6) respectively, and taking care to consider only first order terms in θ and/or β.

The motion equation on the usual configuration space is given by:

$$m\ddot{x}'_i = -\frac{\partial V'}{\partial x'^i} + \left[m\theta_{ij} \left(\frac{\partial^2 V'}{\partial x'_k \partial x'_j} \right) + \beta_{ik} \right] \dot{x}'_k \tag{12}$$

which looks also as the relation (7), but now we are dealing with commutative variables.

4. EXAMPLES OF CLASSICAL SYSTEMS

Let us treat now some examples of classical systems: A free particle ($V(x) = 0$), an harmonic oscillator ($V(x) = \frac{1}{2}Kx^2$), and a gravitational potential ($V(r) = -\frac{K}{r}$).

4.1. Free Particle

In the case of a free classical particle described on the noncommutative CPS, the motion equation (7) reduces to:

$$m\ddot{x}_i = \beta_{ik}\dot{x}_k \Rightarrow m\vec{\gamma} = \vec{v} \wedge \vec{\beta}.$$

This situation looks like the study of the motion of a classical particle of charge q described on the classical phase space in presence of a magnetic field $\vec{B}$:

$$\vec{\beta} = q\vec{B} \tag{13}$$

The quantum analog of this classical system behaves in the same way, such that the gauge invariant velocity operator $\vec{\mathbf{v}}$ that defines the translation operator $U(\vec{a}) = \exp\{i\frac{m}{\hbar}\vec{a}.\vec{\mathbf{v}}\}$ on the noncommutative configuration space do not commute in the sense of (4):

$$[\mathbf{v}_i, \mathbf{v}_j]_\alpha = i\frac{\hbar}{m^2}\epsilon_{ij}^k \beta_k$$

and do not associate:

$$[\mathbf{v}_1, [\mathbf{v}_2, \mathbf{v}_3]_\alpha]_\alpha + [\mathbf{v}_3, [\mathbf{v}_1, \mathbf{v}_2]_\alpha]_\alpha + [\mathbf{v}_2, [\mathbf{v}_3, \mathbf{v}_1]_\alpha]_\alpha = \frac{\hbar^2}{m^3}\vec{\nabla}\cdot\vec{\beta}.$$

This means that the quantum free particle of charge q on a noncommutative phase space looks like the well-known quantum mechanical problem of an ordinary quantum particle moving in the configuration space in presence of a magnetic source, specifically a magnetic monopole. If we do the parallel between the two situations, then this will lead to the interpretation of the presence of a noncommutative perturbation on the phase space as a magnetic source (13). In this framework, the occurring of a nontrivial three cocycle ω_3, (Jackiw, 1985):

$$\omega_3 = -\frac{1}{2\pi\hbar}\int d\vec{r}\,\vec{\nabla}\cdot\vec{\beta}$$

in the usual QM in presence of a magnetic source is deeply tied to a certain topological perturbation of phase space since its triangulation covering at very small scales means that the phase space is no longer commutative.

In the simple case where $\vec{\beta} = \beta\vec{k}$, which means that the noncommutativity is present only on the plane (x,y), this implies a presence of magnetic field in the direction of z-axis and so perturbs the (x,y) plane.

However, within our framework, in the case of a free particle, we have:

$$m\ddot{x}'_i = \beta_{ik}\dot{x}'_k = \epsilon_i^{kl}\dot{x}'_k\beta_l = (\vec{v}' \wedge \vec{\beta})_i = q(\vec{v}' \wedge \vec{B})_i$$

We conclude that a free particle ($\ddot{x}_i = 0$) on the usual CPS is now no longer free on the NCCPS. The noncommutativity on CPS appears to be equivalent to the presence of some magnetic field $\vec{B} = q^{-1}\vec{\beta}$.

4.2. Harmonic Oscillator

Let us consider now the example of an harmonic oscillator characterized by the potential:

$$V(x) = \frac{1}{2}kx^2 = \frac{1}{2}kx_i \star_\alpha x^i$$

In this case, the noncommutative Hamilton's equations (5) and (6) read:

$$\dot{x}_i = \frac{p_i}{m} + k\theta_{ij}x_j, \quad \dot{p}' = -kx_i + \frac{1}{m}\beta_{ij}p_j$$

and the motion equations on the NC configuration space become:

$$m\ddot{x}_i - [\beta + mk\theta]_{ij}\dot{x}_j + kx_i = 0$$

or equivalently:

$$m\vec{\gamma} + \vec{\mu} \wedge \vec{v} + k\vec{x} = \vec{0}$$

where

$$\vec{\mu} = mk\vec{\theta} + \vec{\beta}$$

Investigating these motion equations, one finds that this classical dynamical system on NC configuration space behaves like a harmonic oscillator with the same frequency $\omega_0 = \sqrt{\frac{k}{m}}$, but in the plane perpendicular to the direction of $\vec{\mu}$:

$$\vec{\mu}\cdot[\vec{\gamma} + \omega_0^2\vec{x}] = 0$$

Let's consider, for instance, the simple case where $\vec{\mu} = \mu\vec{k}$, ($\theta_1 = \theta_2 = \beta_1 = \beta_2 = 0$ and $\mu = \mu_3 = \beta_3 + mk\theta_3$). Then, one has:

$$\begin{cases} m\ddot{x}_1 + kx_1 = \mu\dot{x}_2 \\ m\ddot{x}_2 + kx_2 = -\mu\dot{x}_1 \\ m\ddot{x}_3 + kx_3 = 0 \end{cases}$$

The third equation confirms the fact that along the z-axis the system still behaves as a harmonic oscillator with the same frequency. Nevertheless, its motion in the (x, y)-plane is governed by the two first mixed equations. Investigating these two equations, we find:

$$\frac{1}{2}m[\dot{x}_1 \star_\alpha \dot{x}_1 + \dot{x}_2 \star_\alpha \dot{x}_2] + \frac{1}{2}k[x_1 \star_\alpha x_1 + x_2 \star_\alpha x_2]$$
$$= \frac{1}{2}mv^2 + \frac{1}{2}kr^2 = H_{xy} = \mathit{Constant}.$$

This looks like the expression of a conserved Hamiltonian of a planar oscillator.

Finally, we conclude that, in this case, our 3D harmonic oscillator on noncommutative CPS splits into two conservative harmonic oscillators:

$$H = H_{xy} + H_z$$

where

$$H_z = \frac{1}{2}m\dot{x}_3 \star_\alpha \dot{x}_3 + \frac{1}{2}kx_3 \star_\alpha x_3$$

Let us now consider our approach based on considering the primed commutative variables. In this case, the potential is given by:

$$V' = V(x') = \frac{1}{2}kx'^2$$

and we can show that one obtains the same results as before. Nevertheless, let us discuss the correction terms that occur in the new Hamiltonian:

$$H' = H - \frac{1}{2m}\vec{L}\cdot\vec{\mu}$$

This confirms the fact that our 3D harmonic oscillator on noncommutative CPS is equivalent to the usual 3D harmonic oscillator of charge q in presence of some magnetic field:

$$\vec{B} = q^{-1}\vec{\mu}$$

4.3. Gravitational Potential

Let's consider a particle of mass m and charge q moving in a gravitational potential:

$$V(r) = -\frac{k}{r}$$

where $r = \sqrt{x_i \star_\alpha x^i}$. Let's set:

$$\Omega_i = \frac{k}{r^3}\theta_i$$

which we will call the *angular velocity*. Then, the NC Hamilton's equations read:

$$\dot{x}_i = \frac{p_i}{m} + \theta_{ij}\frac{kx^j}{r^3} = \frac{p_i}{m} + (\vec{x} \wedge \vec{\Omega})_i$$

$$\dot{p}_i = -\frac{kx_i}{r^3} + \frac{1}{m}\beta_{ij}p^j = -\frac{kx_i}{r^3} + \frac{1}{m}(\vec{p} \wedge \vec{\beta})_i$$

and the motion equations on the NC configuration space become:

$$m\ddot{x}_i = -\frac{x_i}{r}\frac{k}{r^2} + m\epsilon_i^{jk}(\dot{x}_j\Omega_k + x_j\dot{\Omega}_k) + \epsilon_i^{jk}\dot{x}_j\beta_k$$

or equivalently:

$$m\vec{\gamma} = -\frac{k}{r^2}\frac{\vec{x}}{r} + m(\vec{\dot{x}} \wedge \vec{\Omega} + \vec{x} \wedge \vec{\dot{\Omega}}) + \vec{\dot{x}} \wedge \vec{\beta} = -\frac{k}{r^2}\frac{\vec{x}}{r} + \vec{\dot{x}} \wedge \vec{\sigma} + \vec{x} \wedge \vec{\dot{\sigma}} \quad (14)$$

where

$$\vec{\sigma} = \vec{\beta} + \frac{km}{r^3}\vec{\theta} = \vec{\beta} + m\vec{\Omega}$$

These equations of motion are different from the ones obtained in Romero and Vergara (2003) by a term that comes from the noncommutativity parameter β which is not considered there.

Moreover, it is straightforward to check that the Hamiltonian is a constant of motion:

$$\dot{H} = \frac{1}{2m}[\dot{p}_i \star_\alpha p^i + p_i \star_\alpha \dot{p}^i] + \dot{V}(r) = \frac{1}{m}p_i\dot{p}^i + \frac{k}{r^3}x_i\dot{x}^i = 0$$

and that the components of the angular momentum of this system on NCCPS are no longer conserved:

$$L_i^{\mathrm{NC}} = \epsilon_i^{jk}x_j \star p_k = L_i^{\mathrm{C}} - \frac{mk}{r^3}[\vec{x} \wedge (\vec{x} \wedge \vec{\theta})]_i = L_i^{\mathrm{C}} - m[\vec{x} \wedge (\vec{x} \wedge \vec{\Omega})]_i$$

where:

$$L_i^{\mathrm{C}} = \epsilon_i^{jk}x_j(m\dot{x}_k)$$

is the conserved angular momentum on usual CPS.

Nevertheless, the component along the $\vec{\sigma}$ axis of the angular momentum is conserved:

$$\vec{L}^{\mathrm{NC}} \cdot \vec{\sigma} = \vec{L}^{\mathrm{C}} \cdot \vec{\sigma} = \epsilon^{ijk}\sigma_i x_j(m\dot{x}_k)$$

On the other hand, we remark from (14), that relatively to the $\vec{\sigma}$ axis our system still remains "classical", i.e.:

$$m\vec{\gamma}\cdot\vec{\sigma} = -\frac{k}{r^2}\frac{\vec{x}\cdot\vec{\sigma}}{r} \tag{15}$$

Then, it is more indicated to study the motion of the system in the plane perpendicular to the $\vec{\sigma}$ axis. For this reason, in the following we will consider only one independent noncommutative parameter, namely $\sigma = \sigma_3 = \beta + m\Omega$, with $\theta_1 = \theta_2 = \beta_1 = \beta_2 = 0$ and $\theta = \theta_3$, $\beta = \beta_3$, $\Omega = \Omega_3$. Firstly, along the $\vec{\sigma}$ axis the motion of our system is governed by (See (15)):

$$m\ddot{x}_3 = -\frac{\partial V}{\partial x_3} = -\frac{kx_3}{r^3} \tag{16}$$

Now, let us express the motion equations (14) of this system on the (x,y)-plane in terms of polar coordinates (ρ, ϕ):

$$\begin{cases} m[\ddot{\rho} - \rho\dot{\phi}^2] = -\dfrac{\partial V(\rho)}{\partial\rho} + m\rho\sigma\dot{\phi} = -\dfrac{k}{\rho^2} + m\rho\dot{\phi}\Omega + \rho\dot{\phi}\beta \\ \dfrac{d}{dt}[m\rho^2\dot{\phi}] = -\rho\dfrac{d}{dt}(\rho\sigma) = -m\rho\dfrac{d}{dt}(\rho\Omega) - \beta\rho\dot{\rho} \end{cases} \tag{17}$$

where we have considered the case of equatorial orbits ($\varphi = \frac{\pi}{2} \Rightarrow r = \rho$).

It is easy to check from (17), that the quantity:

$$M = \rho^2(m\dot{\phi} + \sigma) - m\theta V - \frac{\beta}{2}\rho^2 = m\rho^2\dot{\phi} + \frac{2mk\theta}{\rho} + \frac{\beta}{2}\rho^2$$

is a constant of motion since $\dot{M} = 0$.

Returning to the equation (17), we find:

$$m\ddot{\rho} + \frac{k}{\rho^2} - \frac{M^2}{m\rho^3} + \frac{3kM\theta}{\rho^4} = 0$$

where we have neglected second order terms in θ and β.

In order to deduce the trajectory equation $\rho = \rho(\phi)$, let us introduce the following change:

$$u = \frac{1}{\rho}$$

Then, we obtain the following differential equation:

$$[Mu^3 - 4km\theta u^4 - \beta u]\left(\frac{d^2u}{d\phi^2}\right) - [2km\theta u^3 + \beta]\left(\frac{du}{d\phi}\right)^2$$
$$- k\frac{m}{M}u^3 + Mu^4 - 3km\theta u^5 = 0 \tag{18}$$

that differs from the one obtained in Romero and Vergara (2003) by additional terms in β and missing terms of second order in θ and β.

In the classical case, i.e. at the zero order ($\theta = \beta = 0$), we obtain the ordinary Kepler motion equation:

$$\frac{d^2u_0}{d\phi^2} + u_0 = \frac{1}{b}$$

where

$$b = \frac{M^2}{km}.$$

The solution of this equation is given by the elliptic trajectory:

$$u_0 = \frac{1 + e\cos\phi}{b}$$

where e is some parameter representing the eccentricity of the ellipse.

At first order in θ and β, we propose the following solution:

$$u = u_0 + \theta u_1 + \beta u_2 \tag{19}$$

Replacing in (18), one obtains the following differential equations:

$$\begin{cases} \dfrac{d^2u_1}{d\phi^2} + u_1 = F_1(\phi) \\ \dfrac{d^2u_2}{d\phi^2} + u_2 = F_2(\phi) \end{cases}$$

where

$$F_1 = \frac{M}{b^3}\left[2e\cos(\phi) - \frac{3}{2}e^2\cos(2\phi) + \frac{e^2+6}{2}\right]$$

$$F_2 = -\frac{be}{M}\left[\frac{\cos(\phi) + e\cos(2\phi)}{(1+e\cos(\phi))^3}\right]$$

The first differential equation admits the following general solution:

$$u_1(\phi) = \frac{M}{b^3}\left[e\phi\sin(\phi) + \frac{e^2}{2}\cos(2\phi) + \frac{e^2+6}{2}\right]$$

while the second one admits a more complicated general solution which looks like:

$$\begin{aligned} u^2 = -\frac{be}{M}\sin(\phi)\Bigg\{ & A_0\phi\sin(\phi) + A_1\text{arctanh}\left[a\tan\left(\frac{\phi}{2}\right)\right] \\ & + A_2\cot(\phi) + A_3\csc(\phi) + A_4\cot\left(\frac{\phi}{2}\right) + A_5\tan\left(\frac{\phi}{2}\right) \\ & + A_6\cot(\phi)\ln[bu_0] + A_7\left(\frac{A_8\sin(\phi) + A_9\sin(2\phi)}{b^2u_0^2}\right)\Bigg\} \end{aligned}$$

where $A_0 = -\frac{2}{e^2}$ and the other coefficients are functions of e.

Then, to first order in α and β, the general solution of (18) is given by (19), i.e.:

$$u = u_0 + \theta u_1 + \beta u_2 = \frac{1 + e\cos(\phi)}{b} + \left[\frac{Me}{b^3}\theta + \frac{2b}{Me}\beta\right]\phi\sin(\phi)$$
$$+\theta[\ldots] + \beta[\ldots] \approx \left[\frac{1 + e\cos\left[\phi\left(1 - \frac{\xi}{b}\right)\right]}{b}\right] + \ldots$$

The remarkable point is the appearance of terms linear in ϕ in the perturbation terms u_1 and u_2. These interesting terms, that let the original ellipse u_0 change when it precesses, permit us to calculate the possible perihelion shift per revolution due to noncommutativity:

$$\delta\phi_{\mathrm{NC}} = 2\pi\left[\frac{\xi}{b}\right]$$

where:

$$\xi = \frac{M}{b}\theta + \frac{2b^3}{Me^2}\beta$$

Taking into account that:

$$k = mm_s G, \quad b = a(1 - e^2)$$

where m_s is the sun mass and "a" is the average radius of the ellipse, then:

$$\delta\phi_{\mathrm{NC}} = 2\pi\left\{\frac{M}{b^2}\theta + \frac{2b^2}{Me^2}\beta\right\} = 2\pi\left\{\kappa^{\frac{1}{2}}\theta + \frac{2}{e^2}\kappa^{-\frac{1}{2}}\beta\right\}$$

with

$$\kappa = \frac{m^2 m_s G}{a^3(1 - e^2)^3}$$

Furthermore, it has been shown that in the context of General Relativity, the advance of the perihelion with the Schwarzschild metric is given by, (Pireaux *et al.*, 2001).

$$\delta\phi_{\mathrm{RG}} = 2\pi\left\{\frac{3m_s G}{c^2 a(1 - e^2)}\right\}$$

Then, it follows that:

$$\delta\phi_{\mathrm{NC}} = \lambda\delta\phi_{\mathrm{RG}}$$

where

$$\lambda = \frac{a(1 - e^2)c^2}{3Gm_s}\left[\kappa^{\frac{1}{2}}\theta + \frac{2}{e^2}\kappa^{-\frac{1}{2}}\beta\right]$$

In the particular case of the Mercury planet, and using the following data:

$$a \approx 6 \times 10^{10}\text{m}, \quad e \approx 0,2, \quad m \approx 3,3 \times 10^{23}\text{kg}$$

$$m_s \approx 2 \times 10^{30}\text{kg}, \quad G \approx 7 \times 10^{-11}\,\text{m}^3\text{kg}^{-1}\text{s}^{-2}, \quad \hbar \approx 6,6 \times 10^{-34}\,\text{Js}$$

we found that:

$$\kappa \approx 10^{34}\,\text{kg}^2/\text{s}^2, \quad \lambda \approx 1.2 \times 10^7[10^{17}\theta + 50 \times 10^{-17}\beta]$$

and then, the perihelion shift is of order:

$$\delta\phi_{\text{NC}} \approx 2\pi[10^{17}\theta + 50 \times 10^{-17}\beta]$$

Let us recall that the parameters θ and β have been at first considered as perturbation parameters, so they are very small, (Djemai and Smail, 2003). Then, from the above relation, one can deduce that the contribution of the second parameter is very small compared to the one of the first parameter. So, we can ignore it. In this case, our results will be very close to those obtained in Romero and Vergara (2003). In fact, let us evaluate an order of the first parameter by comparing $\delta\phi_{\text{NC}}$ to the experimental data.

Knowing that the observed perihelion shift for Mercury is, (Pireaux *et al.*, 2001):

$$\delta\phi_{\text{obs}} = 2\pi(7.98734 \pm 0.0003) \times 10^{-8}\text{rad/rev}$$

and assuming that $\delta\phi_{\text{NC}} \approx \delta\phi_{\text{obs}}$, it follows that:

$$\theta \approx 8 \times 10^{-25}\text{s/kg}$$

Now, since the noncommutativity effect is considered as a quantum effect of gravity, (Snyder, 1946; Yang, 1947), let us calculate:

$$\sqrt{\hbar\theta} \approx 23 \times 10^{-30}\text{m}.$$

Moreover, General relativity predicts for the perihelion shift:

$$\delta\phi_{\text{RG}} = 2\pi(7.987344) \times 10^{-8}\text{rad/rev}$$

So, we can evaluate a lower bound for θ by means of the difference between the General relativity prediction of the shift and the observed one:

$$|\delta\phi_{\text{NC}}| \leq |\delta\phi_{\text{GR}} - \delta\phi_{\text{obs}}| \approx 4 \times 10^{-14}$$

Then, we get:

$$\begin{aligned}\theta \leq 6 \times 10^{-32} &\rightarrow \hbar \leq 40 \times 10^{-62}\text{m}^2 \\ &\rightarrow \sqrt{\hbar\theta} \leq 63 \times 10^{-32}\text{m} \approx (4 \times 10^4)L_{\text{P}} \\ &\rightarrow \frac{1}{\sqrt{\hbar\theta}} \geq 1.6 \times 10^{-30}\text{m}^{-1}\end{aligned}$$

where L_{P} represents the Planck scale.

Now, let us return to our approach that makes use of primed variables (8). In this framework, the Hamiltonian of our system on NCCPS reads as:

$$H' = H - \frac{1}{2m}\vec{L}^{\mathrm{C}} \cdot \vec{\sigma}$$

From (20), we can interpret the manifestation of noncommutativity on CPS as being equivalent to the presence of some "magnetic field" $\vec{B} = q^{-1}\vec{\sigma}$ that interacts with our system of charge q.

In this frame work, the components of the angular momentum on NCCPS are given by:

$$L_1^{\mathrm{NC}} = L_i' = \epsilon_i^{jk} x_j' p_k' = L_i^{\mathrm{C}} + \frac{1}{2}[\vec{x} \wedge (\vec{x} \wedge \vec{\beta}) - (\vec{p} \wedge \vec{\theta}) \wedge \vec{p}]_i$$

Moreover, it is easy to see that following our framework, we will obtain nearly the same results as described before.

5. CONCLUSION

In this work, I have studied the noncommutative classical mechanics related to the Noncommutative quantum mechanics as described in Djemai and Smail (2003). The same interpretations have been given to the occurrence of noncommutativity effects as in the quantum case. Treating the particular case of a gravitational potential, which is relevant at large scales and which looks like the Coulomb potential at small scales, I show that there is a correction to the perihelion shift of Mercury, and with a parameter $\hbar\theta$ of the order of 10^{-56} m^2 we are in presence of an observable deviation.

Let us remark that the second NC parameter β does not contribute to this correction compared to the contribution of the parameter θ.

Finally, the main point in our work is the fact that the NC parameters which are initially present at a quantum level, occur also at large scales. So, there is a deep link between Physics at small scales and Physics at large scales as it is predicted by UV/IR mixing. This confirm the results obtained in Romero and Vergara (2003).

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